MRC Technical Summary Report # 1921

A HIGHER ORDER GLOBAL APPROXIMATION

METHOD FOR SOLVING AN ABEL INTEGRAL

EQUATION BY QUADRATIC SPLINES
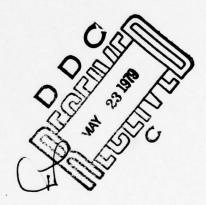
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See 1473 in back.

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February 1979

(Received December 22, 1978)

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MAY 23 1979

Approved for public release

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ABSTRACT

A quadratic spline approximation in C^1 to the solution of the Abel

integral equation

$$\int_0^x \frac{1}{\sqrt{x^2-s^2}}\, y(s)\ ds = f(x)\quad ,\qquad x \geq 0\quad ,$$

is constructed. It is shown that if y''' is Lipschitz continuous, then the

approximation and its first two derivatives converge to the corresponding

exact solutions at each point in the interval of integration, the orders of

convergence being two, two and one, respectively. In addition, if

$y \in C^4[0,1]$, then an asymptotic error estimate for the derivative of the

approximate solution is obtained, and convergence of the approximate solution

is proved to be of order $h^{5/2}$. The method is illustrated by a numerical

example.

SIGNIFICANCE AND EXPLANATION

The significance of the Abel integral equations in physical applications and some description of numerical methods of solution have been discussed in the Significance and Explanation for a recent MRC report [#1904]. In that report, an analysis of a linear spline method of solution was given.

In the present paper, a global approximation method is developed for obtaining higher accuracy results involving higher order splines with full continuity. The quadratic spline case is investigated in detail. The technique is to differentiate the original equation, and solve the differentiated equation by using a quadratic spline in $C_u^{(1)}$. The computational effort required is only marginally greater than that required for a linear spline solution of the original equation. Convergence is obtained not only for the approximate solution but also for its first two derivatives at each point in the interval of integration.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

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1. __Introduction__. A direct numerical method using quadratic spline of class C^1 will be presented and analyzed for solving the Abel integral equation:

$$(1.1) \qquad \int_0^x \frac{1}{\sqrt{x^2-s^2}}\, y(s)\, ds = f(x) \quad , \quad x \geq 0 \quad .$$

This is a classical equation, and it occurs very often in applications (see, for example, Noble [13] and Anderssen [1]). It is well known that this equation can be converted to give

$$(1.2) \qquad y(x) = \frac{2}{\pi} \frac{d}{dx} \int_0^x \frac{s f(s)}{\sqrt{x^2-s^2}}\, ds \quad ,$$

$$= \frac{2}{\pi} \left\{ f(0) + x \int_0^x \frac{f'(s)}{\sqrt{x^2-s^2}}\, ds \right\} \quad , \quad x \geq 0 \quad .$$

Methods based on the numerical evaluation of this inversion formula have been considered in [6] and [11]. They all have to deal with the presence of a derivative in the inversion formula, and are inapplicable to equations for which an inversion formula is not known. Direct methods for solving equation (1.1) and related equations have been proposed and studied by a number of authors [1-5, 7, 8, 11, 12, 14, 15].

Recently, step by step methods based on the concept of product integration developed by Young [16] for the solution of (1.1) by using linear splines have been considered and analyzed by several authors ([2], [3], [8]). Atkinson [2] gives a convergence theorem but does not prove that the convergence is $O(h^2)$. Benson [3] obtains $O(h^2)$ and also derives an asymptotic formula for the discretization error, but his method depends on a complicated analysis of product integration. Hung [8] obtains similar results as Benson [3] under slightly weaker conditions and by a much simpler method.

It might be supposed that the use of splines of order greater than one would result in a higher order approximation method for equation (1.1). Such an approach, however, fails, because of instability, in the case that the spline is of full continuity and the approximation to the solution is based on using equation (1.1) alone (see Appendix). The divergence of higher order spline methods arise from the strictness of the continuity requirements on the spline functions. It is possible to generate higher order methods by relaxing the continuity requirements on the spline functions, and this is the type of methods considered by Brunner and Linz. Brunner [4, 5] uses piecewise polynomials of a given degree and of class C to generate approximate solutions for equation (1.1) as well as equations of a similar type, and shows that convergence of order m is obtained if m^{th} degree piecewise polynomials are used. His method is based on using the original equation alone, and is basically a block by block method in the sense of [10]. Linz [11] shows numerically that a fourth order Hermite interpolation method can be constructed for equation (1.1) by using cubic spline in C^1. His method is based on using the differentiated form of equation (1.1) simultaneously with equation (1.1).

In this paper we show that the divergence of higher order splines with full continuity depends upon the particular method used, and is not necessarily a result of strictness of the continuity requirements. We develop a global approximate solution for equation (1.1) by quadratic splines in C^1. The method to be described in Section 2 is self-starting and is based on using the differentiated form of (1.1) alone. In Section 3, we prove convergence for the approximate solution as well as its first two derivatives. A simple asymptotic error formula is derived in Section 4 for the derivative of the approximate solution by which convergence of the approximate solution itself is shown to be of order $h^{5/2}$. Finally, a numerical example is presented in Section 5.

In this paper, it is assumed that $0 \le x \le 1$, but this restriction is not essential.

2. The Quadratic Spline Method. Let $x_i = ih$, $i = 0, 1, 2, \cdots$, where h is an arbitrary constant step size. Let $y(x)$ be the exact solution of (1.1), and let Y_i, Y'_i denote approximations to $y(x_i)$, $y'(x_i)$, respectively. We use a quadratic spline $P(x)$ in $C^1[0,1]$

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with knots at the points x_i as an approximation to $y(x)$, i.e., for $i = 0, 1, 2, \cdots$

$$(2.1) \qquad P(x) = Y_i + \frac{h}{2} [(2u_i(x) - u_i^2(x))Y_i' + u_i^2(x)Y_{i+1}'] \quad , \quad x_i \le x \le x_{i+1} \quad ,$$

where $u_i(x) = (x-x_i)/h$. Differentiating (2.1) we obtain the derivative of $P(x)$,

$$(2.2) \qquad P'(x) = u_i(x)Y_{i+1}' - u_{i+1}(x)Y_i' \quad , \quad x_i \le x \le x_{i+1} \quad .$$

Both $P(x)$ and $P'(x)$ are continuous at the knots.

In order to obtain a numerical solution of (1.1) by using the quadratic spline (2.1), we have to introduce the differentiated form of (1.1). Assume that we can differentiate (1.1). Then

$$(2.3) \qquad \int_0^x \frac{s}{\sqrt{x^2-s^2}} y'(s) \, ds = xf'(x) \quad , \quad x \ge 0 \quad .$$

The approximate solution $P(x)$ of the integral equation (1.1) is derived from values $P(x_i) = Y_i$ and $P'(x_i) = Y_i'$. We know how to find Y_{k+1}', assuming that we know Y_i', $i = 0, 1, \cdots, k$. We can then deduce Y_{k+1} from the equation obtained by setting $x = x_{k+1}$ in (2.1), which gives

$$(2.4) \qquad Y_{k+1} = Y_k + \frac{h}{2} (Y_k' + Y_{k+1}') \quad .$$

To compute Y_i', we require that $P'(x)$ satisfies (2.3), i.e., $y'(x)$ is replaced by $P'(x)$ derived from the values $P'(x_i) = Y_i'$ computed from

$$(2.5) \qquad \int_0^{x_k} \frac{s}{\sqrt{x_k^2-s^2}} P'(s) \, ds = x_k f'(x_k) \quad .$$

This can be rewritten in the form:

$$(2.6) \qquad \sum_{i=0}^k w_{k,i} Y_i' = x_k f'(x_k) \quad , \quad k = 1, 2, \cdots \quad ,$$

where

$$(2.7) \quad \begin{cases} w_{k,0} = -\int_{x_0}^{x_1} \dfrac{s}{\sqrt{x_k^2 - s^2}}\, u_1(s)\, ds \quad, \\[3mm] w_{k,i} = \int_{x_{i-1}}^{x_i} \dfrac{s}{\sqrt{x_k^2 - s^2}}\, u_{i-1}(s)\, ds - \int_{x_i}^{x_{i+1}} \dfrac{s}{\sqrt{x_k^2 - s^2}}\, u_{i+1}(s)\, ds \quad, \quad i = 1, \cdots, k-1 \quad, \\[3mm] w_{k,k} = \int_{x_{k-1}}^{x_k} \dfrac{s}{\sqrt{x_k^2 - s^2}}\, u_{k-1}(s)\, ds \quad. \end{cases}$$

Equation (2.6) is a nonsingular triangular system for Y_i', because

$$(2.8) \qquad \left(\frac{4}{15}\right)\frac{5k-1}{\sqrt{2k}} \cdot h \le w_{k,k} \le \left(\frac{4}{15}\right)\frac{5k-1}{\sqrt{2k-1}} \cdot h$$

for $k = 1, 2, \cdots$.

Since $y(0) = \frac{2}{\pi} f(0)$ and $y'(0) = f'(0)$, we take $Y_0 = y(0)$ and $Y_0' = y'(0)$. The values Y_i', Y_i ($i = 1, 2, \cdots$) can then be determined successively by (2.6) together with (2.4). Note that this approach requires no starting procedure.

An estimate of $y''(x)$ is given by the second derivative of (2.1). If we denote this (constant) estimate of $y''(x)$ in $x_i \le x < x_{i+1}$ by Y_i'', this gives

$$(2.9) \qquad Y_i'' = \frac{1}{h}(Y_{i+1}' - Y_i') \quad, \qquad x_i \le x < x_{i+1} \quad.$$

3. Convergence Results. The proofs of Theorem 3.1 and Theorem 4.1 require the following lemma which is a consequence of the standard results for regular infinite systems of algebraic equations by Kantorovich and Krylov in [7, p.27].

Lemma 3.1. If there exist a constant $C > 0$ and an integer $N \ge 1$, all independent of k, such that

$$|x_i| \le C \quad, \qquad\qquad\qquad\qquad i = 0, 1, \cdots, N \quad,$$

$$|x_{k+1}| \le \sum_{i=0}^{k} |\alpha_{k+1,i}|\, |x_i| + |\beta_k| \quad, \qquad k = N, N+1, \cdots \quad,$$

with

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$$\rho_k = 1 - \sum_{i=0}^{k} |\alpha_{k+1,i}| > 0 \quad ,$$

$$|\beta_k| \leq C\rho_k \quad , \qquad\qquad k = N, N + 1, \cdots \ ,$$

then $\qquad\qquad |x_i| \leq C \quad , \qquad\qquad i = 0, 1, 2, \cdots \ .$

The proof of this lemma can be found in [8, p.3].

Let $y(x)$ be the exact solution of (1.1) and define the discretization error function $\epsilon(x)$ by $\epsilon(x) = y(x) - p(x)$, where $p(x)$ is the quadratic spline approximation to $y(x)$ obtained from our numerical method. Denote $\epsilon^{(r)}(x_i)$ by $\epsilon_i^{(r)}$ $(r = 0, 1, 2)$. We state the following theorem:

<u>Theorem 3.1</u>. If $y'''(x)$ is Lipschitz continuous on $[0,1]$, then

$$(3.1) \qquad\qquad |\epsilon_k'| = 0(h^2) \quad , \qquad k = 1, 2, \cdots \ .$$

<u>Proof</u>: Applying Taylor's theorem, it is not difficult to show

$$(3.2) \quad y(x) = y(x_i) + \frac{h}{2} [(2u_i(x) - u_i^2(x))y'(x_i) + u_i^2(x)y'(x_{i+1})] + \varphi(x) \quad , \qquad x_i \leq x \leq x_{i+1} \ ,$$

where

$$(3.3) \qquad\qquad \varphi(x) = \frac{h^2}{2} \left[\int_{x_i}^{x} \left(\frac{x-s}{h}\right)^2 y'''(s) \ ds + u_i^2(x) \int_{x_i}^{x_{i+1}} u_{i+1}(s)y'''(s) \ ds \right] \ ,$$

with $u_i(x)$ as defined in (2.1).

Subtracting (2.1) from (3.2) gives

$$(3.4) \quad \epsilon(x) = \epsilon_i + \frac{h}{2} [(2u_i(x) - u_i^2(x))\epsilon_i' + u_i^2(x)\epsilon_{i+1}'] + \varphi(x) \quad , \qquad x_i \leq x \leq x_{i+1} \ .$$

Differentiating (3.4), we have the derivative of $\epsilon(x)$,

$$(3.5) \qquad \epsilon'(x) = [u_i(x)\epsilon_{i+1}' - u_{i+1}(x)\epsilon_i'] + \varphi'(x) \quad , \qquad x_i \leq x \leq x_{i+1} \ ,$$

where $\varphi'(x)$ can be represented in Lagrange form as

$$(3.6) \qquad\qquad \varphi'(x) = \frac{h^2}{2} y'''(\eta_i(x))u_i(x)u_{i+1}(x) \quad , \qquad x_i \leq \eta_i(x) \leq x_{i+1} \ .$$

Since both $y'(x)$ and $P'(x)$ satisfy (2.3) at each knot $x = x_k$,

$$(3.7) \qquad\qquad \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \frac{s}{\sqrt{x_k^2 - s^2}} \ \epsilon'(s) \ ds = 0 \quad , \qquad k = 1, 2, \cdots \ .$$

This can be rewritten as

$$(3.8) \qquad \sum_{i=0}^{k} w_{k,i} \epsilon_i' = R_k \quad , \qquad k = 1, 2, \cdots \quad ,$$

where

$$(3.9) \qquad R_k = -\sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \frac{s}{\sqrt{x_k^2 - s^2}} \, \varphi'(s) \, ds \quad ,$$

and the $w_{k,i}$'s are defined in (2.7).

Multiply (3.8) by k, difference the resulting equation for k and $k + 1$, and then divide by $(k + 1)w_{k+1,k+1}$ to yield the required error equation

$$(3.10) \qquad \epsilon_{k+1}' = \sum_{i=0}^{k} a_{k+1,i} \epsilon_i' + b_k \quad , \qquad k = 1, 2, \cdots \quad ,$$

where

$$(3.11) \qquad a_{k+1,i} = \frac{k w_{k,i} - (k+1) w_{k+1,i}}{(k+1) w_{k+1,k+1}} \quad , \qquad i = 0, 1, \cdots, k \quad ,$$

and

$$(3.12) \qquad b_k = \frac{(k+1) R_{k+1} - k R_k}{(k+1) w_{k+1,k+1}} \quad , \qquad k = 1, 2, \cdots \quad .$$

Note that (3.10) is not the only possible error equation that can be constructed for the proof of convergence. For instance, we can simply difference equation (3.8) for k and $k + 1$. However, the procedure used here simplifies the asymptotic error analysis in Section 4.

Equation (3.10) implies that

$$(3.13) \qquad |\epsilon_{k+1}'| \leq \sum_{i=0}^{k} |a_{k+1,i}| \, |\epsilon_i'| + |b_k| \quad , \qquad k = 1, 2, \cdots \quad .$$

Since $\epsilon_0' = 0$, it is easily shown from equation (3.8) by using Lemma 3.3(a) below that $\epsilon_i' = 0(h^2)$ for $i = 1, \cdots, K$, with K as defined in Lemma 3.2 below. On the basis of this together with Lemma 3.2(b) and Lemma 3.3(c) below we can apply Lemma 3.1 on (3.13) and obtain (3.1). Hence the proof of Theorem 3.1 is completed.

Theorem 3.2. Let the assumption of Theorem 3.1 be satisfied, then for $k = 1, 2, \cdots$

$$(3.14) \qquad |\epsilon_k| = 0(h^2) \quad .$$

Proof: Setting $x = x_{i+1}$ in (3.4) and noting that $\varphi(x_{i+1}) = 0(h^3)$, we obtain

$$(3.15) \qquad \epsilon_{i+1} = \epsilon_i + \frac{h}{2} (\epsilon_i' + \epsilon_{i+1}') + 0(h^3) \quad .$$

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Since $\epsilon_0 = 0$, by means of (3.1), (3.14) immediately follows from (3.15) by induction.

In the following theorem we can show that the approximate solution $P(x)$ along with its first and second derivatives converges to the corresponding exact solutions at each point in the interval of integration $[0,1]$.

Theorem 3.3. Let the assumption of Theorem 3.1 be satisfied, then for any fixed $x \in [0,1]$,

$$|\epsilon(x)| = 0(h^2) \quad , \quad |\epsilon'(x)| = 0(h^2) \quad , \quad |\epsilon''(x)| = 0(h) \quad .$$

Proof: By means of (3.1) and (3.14), Theorem 3.3 follows immediately from equation (3.4), (3.5) and the equation resulting from differentiating (3.4) twice.

Lemma 3.2. If the $a_{k+1,i}$'s are defined by (3.11), then there exists an integer $k \geq 1$, independent of h and k, such that

$$\text{(a)} \quad a_{k+1,i} \geq 0 \quad , \qquad\qquad i = 0, 1, \cdots, k - 1, \ k \geq 1 \ ,$$

$$\qquad\quad a_{k+1,k} \geq 0 \quad , \qquad\qquad k \geq K \ ,$$

$$\text{(b)} \quad 1 - \sum_{i=0}^{k} |a_{k+1,i}| \geq \frac{3}{2} \frac{1}{\sqrt{k+1}} \quad , \qquad k \geq K \ .$$

Proof of (a): From (3.11), using (2.7) and (2.8) we have

$$a_{k+1,0} = \frac{1}{(k+1)w_{k+1,k+1}} \int_{x_0}^{x_1} \left[\frac{1}{\sqrt{1-\left(\frac{s}{x_k}\right)^2}} - \frac{1}{\sqrt{1-\left(\frac{s}{x_{k+1}}\right)^2}} \right] \frac{s(x_1-s)}{h^2} ds \geq 0 \quad ,$$

since the integrand is non-negative. Similarly, we can prove that $a_{k+1,i} \geq 0$ for $i = 1$, $\cdots, k - 1$. Finally, it is easy to show that $a_{k+1,k}$ tends to $3 - 2\sqrt{2}$ as k increases, therefore there exists an integer K, independent of h and k such that $a_{k+1,k} \geq 0$ for $k \geq K$.

Proof of (b): Since by using (2.7) we obtain

$$(3.16) \qquad\qquad \sum_{i=0}^{k} w_{k,i} = \int_{0}^{x_k} \frac{s}{\sqrt{x_k^2-s^2}} ds = x_k$$

for $k = 1, 2, \cdots$, it can easily be shown that for $k = 1, 2, \cdots$

$$(3.17) \qquad\qquad \sum_{i=0}^{k} a_{k+1,i} = 1 - \frac{(2k+1)h}{(k+1)w_{k+1,k+1}} \leq 1 - \frac{3}{2} \frac{1}{\sqrt{k+1}} \quad .$$

By means of part (a) and (3.17), the result of part (b) immediately follows.

Lemma 3.3. If $y'''(x)$ is Lipschitz continuous on $[0,1]$, then there exist constants C_1, C_2, $C_3 > 0$, independent of h and k, such that

$$\text{(a)} \quad |R_k| \le C_1 k h^3 \quad,$$

$$\text{(b)} \quad |R_{k+1} - R_k| \le C_2 h^3 \quad,$$

$$\text{(c)} \quad |b_k| \le C_3 \frac{h^2}{\sqrt{k}} \quad,$$

for $k = 1, 2, \cdots$, where R_k and b_k are defined in (3.9) and (3.12), respectively.

Proof of (a): Let $M_3 = \max\limits_{x \in [0,1]} |y'''(x)|$. Then by straightforward estimation, it follows from (3.9) that

$$(3.18) \qquad |R_k| \le C_1 k h^3 \quad, \quad k = 1, 2, \cdots \quad,$$

where $C_1 = \frac{1}{8} M_3$.

Proof of (b): Subtraction of (3.9) from (3.9) with k replaced by $k + 1$, and by a change of the variable of integration, it is not difficult to show that

$$(3.19) \qquad R_{k+1} - R_k = A_k^{(1)} + A_k^{(2)} + A_k^{(3)} + A_k^{(4)} \quad, \quad k = 1, 2, \cdots \quad,$$

where

$$A_k^{(1)} = - \frac{h^2}{2} \sum_{i=0}^{k-1} \int_{x_{i+1}}^{x_{i+2}} \frac{s}{\sqrt{x_{k+1}^2 - s^2}} \, [y'''(\eta_{i+1}(s)) - y'''(\eta_i(s-h))] \, u_{i+1}(s) u_{i+2}(s) \, ds \quad,$$

$$A_k^{(2)} = - \frac{h^2}{2} \sum_{i=0}^{k-1} \int_{x_{i+1}}^{x_{i+2}} \left[\frac{1}{\sqrt{x_{k+1}+s}} - \frac{1}{\sqrt{x_{k-1}+s}} \right] \frac{s}{\sqrt{x_{k+1}-s}} \, y'''(\eta_i(s-h)) u_{i+1}(s) u_{i+2}(s) \, ds \quad,$$

$$A_k^{(3)} = - \frac{h^3}{2} \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \frac{s}{\sqrt{x_k^2 - s^2}} \, y'''(\eta_i(s)) u_i(s) u_{i+1}(s) \, ds \quad,$$

$$A_k^{(4)} = - \frac{h^2}{2} \int_{x_0}^{x_1} \frac{s}{\sqrt{x_{k+1}^2 - s^2}} \, y'''(\eta_0(s)) u_0(s) u_1(s) \, ds \quad.$$

Let L_3 be the Lipschitz constant for y'''. Then by straightforward estimation and noting that $hk \leq 1$, we obtain from (3.19)

$$(3.20) \qquad |R_{k+1} - R_k| \leq \frac{1}{4} L_3 h^3 + \frac{1}{2} M_3 h^3 + \frac{1}{8} M_3 k h^4 + \frac{1}{8} M_3 \frac{h^3}{k} \leq C_2 h^3 \quad ,$$

where $C_2 = \frac{1}{4} (L_3 + 3M_3)$.

Proof of (c): From (3.12), using (3.18), (3.20) and (2.8), it can easily be shown that for $k = 1, 2, \cdots,$

$$|b_k| \leq \frac{|R_{k+1} - R_k| + \frac{1}{k+1} |R_k|}{w_{k+1,k+1}} \leq C_3 \frac{h^2}{\sqrt{k}} \quad ,$$

where $C_3 = \frac{3}{2} (C_1 + C_2)$.

4. An Asymptotic Error Formula. In this section, we obtain, under the assumption that $y \in C^4[0,1]$, an asymptotic error expression for ϵ'_k which is essential in proving that convergence of the approximate solution is of order $h^{5/2}$.

Theorem 4.1. If $y \in C^4[0,1]$, then for $k = 1, 2, \cdots$

$$(4.1) \qquad \qquad \epsilon'_k = \frac{h^2}{12} y'''(x_k) + 0(h^2/\sqrt{k}) \quad .$$

Proof: Since by assumption $y(x) \in C^4[0,1]$, it is not difficult to show that, for $x \in [x_i, x_{i+1}]$

$$(4.2) \quad \epsilon(x) = \epsilon_i + \frac{h}{2} [(2u_i(x) - u_i^2(x))\epsilon'_i + u_i^2(x)\epsilon'_{i+1}] + \frac{h^3}{12} y'''(x_i) [2u_i^3(x) - 3u_i^2(x)] + \rho(x) \quad ,$$

where

$$\rho(x) = \frac{h^3}{12} [2 \int_{x_i}^{x} \left(\frac{x-s}{h}\right)^3 y^{(4)}(s) \, ds - 3u_i^2(x) \int_{x_i}^{x_{i+1}} u_{i+1}^2(s) y^{(4)}(s) \, ds] \quad ,$$

with $u_i(x)$ as defined in (2.1).

Differentiating (4.2) we obtain the derivative of $\epsilon(x)$,

$$(4.3) \quad \epsilon'(x) = [u_i(x)\epsilon'_{i+1} - u_{i+1}(x)\epsilon'_i] + \frac{h^2}{2} y'''(x_i) u_i(x) u_{i+1}(x) + \rho'(x) \quad , \qquad x_i \leq x \leq x_{i+1} \quad ,$$

where $\rho'(x)$ can be represented in Lagrange form as

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(4.4) $\rho'(x) = \frac{h^3}{6} [u_i^3(x)y^{(4)}(\eta(x)) - u_i(x)y^{(4)}(\eta(x_{i+1}))]$, $x_i \le \eta(x) \le x_{i+1}$.

Adding and subtracting $-\frac{h^2}{12} [u_i(x)y'''(x_{i+1}) - u_{i+1}(x)y'''(x_i)]$ to the right-hand side of (4.3) and using the Taylor series expansion for $y'''(x)$ at x_i for $x = x_{i+1}$, it is not difficult to show that

(4.5) $\epsilon'(x) = [u_i(x)(\epsilon'_{i+1} - \frac{h^2}{12} y'''(x_{i+1})) - u_{i+1}(x)(\epsilon'_i - \frac{h^2}{12} y'''(x_i))] + \Phi(x)$, $x_i \le x \le x_{i+1}$,

where

$$\Phi(x) = \frac{h^2}{2} y'''(x_i)[u_i(x)u_{i+1}(x) + \frac{1}{6}] + [\rho'(x) + \frac{h^3}{12} u_i(x)y^{(4)}(\xi(x_{i+1}))]$$, $x_i \le \xi(x) \le x_{i+1}$.

By substituting (4.5) into (3.7) we obtain

(4.6) $$\sum_{i=0}^{k} w_{k,i}(\epsilon'_i - \frac{h^2}{12} y'''(x_i)) = \hat{R}_k$$, $k = 1, 2, \cdots$,

where

(4.7) $$\hat{R}_k = -\sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \frac{s}{\sqrt{x_k^2 - s^2}} \Phi(s) \, ds$$.

Comparing (4.6) and (3.8) we note that $\hat{R}_k$ is converging faster than R_k. If we derive an error equation in the same way as in the proof of convergence in Section 3 without modifying (4.6) we expect the parts in the derived error equation which correspond to β_k and ρ_k in Lemma 3.1 would have unbalanced rates of convergence, with the former converging faster than the latter. By applying Lemma 3.1 to such an equation we will fail to obtain the result we expect. To avoid this, we define $\hat{\epsilon}'_i = \sqrt{i+1}(\epsilon'_i - \frac{h^2}{12} y'''(x_i))$ and rewrite (4.6) as

(4.8) $$\sum_{i=0}^{k} w_{k,i} \frac{1}{\sqrt{i+1}} \hat{\epsilon}'_i = \hat{R}_k$$, $k = 1, 2, \cdots$.

Now multiply (4.8) by k, difference the resulting equation for k and $k + 1$, then divide by $(k + 1)w_{k+1,k+1}/\sqrt{k+2}$ to yield the required error equation:

(4.9) $$\hat{\epsilon}'_{k+1} = \sum_{i=0}^{k} \hat{a}_{k+1,i}\hat{\epsilon}'_i + \hat{b}_k$$, $k = 1, 2, \cdots$,

where

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(4.10)
$$\hat{a}_{k+1,i} = \frac{\sqrt{k+2}}{\sqrt{i+1}} \, a_{k+1,i} \quad , \qquad i = 0, 1, \cdots, k \quad ,$$

and

(4.11)
$$\hat{b}_k = \frac{\sqrt{k+2}}{(k+1)w_{k+1,k+1}} \, [(k+1)\hat{R}_{k+1} - k\hat{R}_k] \quad , \qquad k = 1, 2, \cdots \quad ,$$

with $a_{k+1,i}$ as defined by (3.11).

Equation (4.9) implies that

(4.12)
$$|\hat{\epsilon}'_{k+1}| \le \sum_{i=0}^{k} |\hat{a}_{k+1,i}| \, |\hat{\epsilon}'_i| + |\hat{b}_k| \quad , \qquad k = 1, 2, \cdots \quad .$$

Since $\hat{\epsilon}'_0 = \epsilon'_0 - \frac{h^2}{12} y'''(x_0) = O(h^2)$, it is easily shown from equation (4.8) by using Lemma 4.2(a) below that $\hat{\epsilon}'_i = O(h^2)$ for $i = 1, \cdots, \hat{K}$, with $\hat{K}$ as defined in Lemma 4.1 below. On the basis of this together with Lemma 4.1(b) and Lemma 4.2(c) below we can apply Lemma 3.1 on (4.12) and conclude that

$$\hat{\epsilon}'_k = \sqrt{k+1}\,(\epsilon'_k - \frac{h^2}{12} y'''(x_k)) = O(h^2) \quad , \qquad k = 0, 1, 2, \cdots \quad .$$

Thus the proof of Theorem 4.1 is completed.

__Theorem 4.2.__ Let the assumption of Theorem 4.1 be satisfied, then for $k = 1, 2, \cdots$

(4.13)
$$|\epsilon_k| = O(\sqrt{k}\,h^3) \quad .$$

__Proof:__ Setting $x = x_{i+1}$ in (4.2) and noting that $\rho(x_{i+1}) = O(h^4)$, we obtain

(4.14)
$$\epsilon_{i+1} = \epsilon_i + \frac{h}{2}(\epsilon'_i + \epsilon'_{i+1}) - \frac{h^3}{12} y'''(x_i) + O(h^4) \quad .$$

Since $\epsilon_0 = 0$, by means of (4.1) and using the fact that $\sum_{i=1}^{k} \frac{1}{\sqrt{i}} \le 2\sqrt{k}$ for $k = 1, 2, \cdots$, (4.13) follows immediately from (4.14) by induction.

__Theorem 4.3.__ Let the assumption of Theorem 4.1 be satisfied, then for any fixed $x \in [0,1]$

$$|\epsilon(x)| = O(h^{5/2}) \quad .$$

__Proof:__ By means of (4.1) and (4.13), Theorem 4.3 follows immediately from equation (4.2).

__Lemma 4.1.__ If the $\hat{a}_{k+1,i}$'s are defined by (4.10), then there exist integers K and $\hat{K} \ge 1$, independent of h and k, such that

(a) $\hat{a}_{k+1,i} \geq 0$, $\qquad\qquad\qquad$ $i = 0, 1, \cdots, k - 1, k \geq 1$,

$\qquad$ $\hat{a}_{k+1,k} \geq 0$, $\qquad\qquad\qquad$ $k \geq K$,

(b) $1 - \sum_{i=0}^{k} |\hat{a}_{k+1,i}| \geq \dfrac{1-C}{\sqrt{k+1}}$, $\quad k \geq \hat{K}$,

for an appropriate C, $0 \leq C < 1$.

<u>Proof of (a)</u>: By means of Lemma 3.2(a), part (a) follows immediately from (4.10).

<u>Proof of (b)</u>: Since it would be very complex if we estimate $\sum_{i=0}^{k} \hat{a}_{k+1,i}$ directly, we introduce $\tilde{a}_{k+1,i}$:

(4.15) $\quad \tilde{a}_{k+1,i} = a_{k+1,i} + \dfrac{\left[1 + C(1 + \frac{1}{k})\right] w_{k,i}}{(k+1)w_{k+1,k+1}}$, $\quad 0 \leq C < 1, i = 0, 1, \cdots, k$.

By using Lemma 3.2(a) and noting the non-negativity of the $w_{k,i}$ from (2.7), we obtain

(4.16) $\qquad\qquad$ $\tilde{a}_{k+1,i} \geq 0$, $\quad i = 0, 1, \cdots, k - 1, k \geq 1$,

$\qquad\qquad$ $\tilde{a}_{k+1,k} \geq 0$, $\quad k \geq K$,

where K is the same K as defined in Lemma 3.2.

$\qquad$ Using (3.16) and (4.16) it can easily be verified that

(4.17) $\qquad\qquad$ $1 - \sum_{i=0}^{k} |\tilde{a}_{k+1,i}| \geq \dfrac{1-C}{\sqrt{k+1}}$, $\quad k \geq K$.

If we can show that for an appropriate C, $0 \leq C < 1$, $\tilde{a}_{k+1,i} - \hat{a}_{k+1,i} \geq 0$, then the proof is completed. To do this, it is sufficient to show that for $i = 0, 1, \cdots, k$,

$$D_{k+1,i} = \left[1 + C(1 + \frac{1}{k})\right] w_{k,i} - \left[\dfrac{\sqrt{k+2}}{\sqrt{i+1}} - 1\right] \left[kw_{k,i} - (k+1)w_{k+1,i}\right] \geq 0 \ .$$

By using (2.7) and letting $s = ht$, it is easily verified that for $i = 1, \cdots, k - 2$,

(4.18) $\qquad$ $D_{k+1,i} \geq \int_{i-1}^{i} L_{k,i}(t)\, t(t-i+1)\, dt + \int_{i}^{i+1} L_{k,i}(t)\, t(i+1-t)\, dt$,

where

$$L_{k,i}(t) = \dfrac{1 + C(1 + \frac{1}{k})}{\sqrt{k^2-t^2}} - \left(\dfrac{\sqrt{k+2}}{\sqrt{i+1}} - 1\right) \dfrac{t^2}{\left(\sqrt{k^2-t^2}\right)^3} \ .$$

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Since for $i - 1 \le t \le i + 1$, $i = 1, \cdots, k - 2$, and $0 \le C < 1$

$$L_{k,i}(t) \ge \frac{C(1 + \frac{1}{k})\left[k^2 - (i+1)^2\right] + \left[k^2 - \sqrt{k+2}\,(i+1)^{3/2}\right]}{\left(\sqrt{k^2 - (i-1)^2}\right)^3} \ge 0 \ ,$$

it is obvious from (4.18) that $D_{k+1,i} \ge 0$ for $i = 1, \cdots, k - 2$. For simplicity, take $C = 0$. Then with this value of C, it is not difficult to show that as k increases $D_{k+1,i}$ $(i = 0, k - 1, k)$ tends to $\frac{1}{6k}h$, $\frac{2\sqrt{2k}}{3}(3\sqrt{3} - 4\sqrt{2} + 1)h$, and $\frac{\sqrt{2k}}{3}(2\sqrt{2} - 1)h$, respectively, and are therefore greater than zero for sufficiently large k.

Thus for $i = 0, 1, \cdots, k$, there exists an integer $\tilde{K} \ge 1$, independent of h and k, such that

$$\tilde{a}_{k+1,i} \ge \hat{a}_{k+1,i} \ , \qquad k \ge \tilde{K} \ .$$

By means of part (a), (4.16) and (4.17), it follows that

$$1 - \sum_{i=0}^{k} |\hat{a}_{k+1,i}| \ge 1 - \sum_{i=0}^{k} |\tilde{a}_{k+1,i}| \ge \frac{1-C}{\sqrt{k+1}} \ ,$$

for $C = 0$, and $k \ge \hat{K}$, with $\hat{K} = \max(K, \tilde{K})$.

Lemma 4.2. If $y \in C^4[0,1]$, then there exist constants $\hat{C}_1$, $\hat{C}_2$, $\hat{C}_3 > 0$, independent of h and k, such that

$$\text{(a)} \quad |\hat{R}_k| \le \hat{C}_1 \sqrt{k}\, h^3 \ ,$$

$$\text{(b)} \quad |\hat{R}_{k+1} - \hat{R}_k| \le \hat{C}_2 \frac{h^3}{\sqrt{k}} \ ,$$

$$\text{(c)} \quad |\hat{b}_k| \le \hat{C}_3 \frac{h^2}{\sqrt{k}} \ ,$$

for $k = 1, 2, \cdots$, where $\hat{R}_k$ and $\hat{b}_k$ are defined in (4.7) and (4.11), respectively.

Proof of (a): By repeated integration by parts, it is not difficult to show that for $i = 0, 1, \cdots, k - 1$

$$(4.19) \qquad \frac{h^2}{2} \int_{x_i}^{x_{i+1}} \frac{s}{\sqrt{x_k^2 - s^2}} y'''(x_i)\left[u_i(s)u_{i+1}(s) + \frac{1}{6}\right] ds$$

$$= \frac{h^4}{8} \int_{x_i}^{x_{i+1}} \frac{x_k^2 s}{\left(\sqrt{x_k^2 - s^2}\right)^5} y'''(x_i)u_i^2(s)u_{i+1}^2(s)\, ds \ .$$

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Using (4.19) we can rewrite (4.7) as

$$(4.20) \qquad \hat{R}_k = \hat{A}_k^{(1)} + \hat{A}_k^{(2)} \quad , \qquad k = 1, 2, \cdots \, ,$$

where

$$\hat{A}_k^{(1)} = - \frac{h^4}{8} \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \frac{x_k^2 s}{\left(\sqrt{x_k^2 - s^2}\right)^5} y'''(x_i) u_i^2(s) u_{i+1}^2(s) \, ds \quad ,$$

$$\hat{A}_k^{(2)} = - \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \frac{s}{\sqrt{x_k^2 - s^2}} [\rho'(s) + \frac{h^3}{12} u_i(s) y^{(4)}(\xi(x_{i+1}))] \, ds \quad ,$$

with $\rho'(s)$ as defined in (4.4).

Let $M_3 = \max\limits_{x \, \in \, [0,1]} |y'''(x)|$ and $M_4 = \max\limits_{x \, \in \, [0,1]} |y^{(4)}(x)|$. Then by straightforward estimation, and noting that $hk \leq 1$, we obtain from (4.20)

$$(4.21) \qquad |\hat{R}_k| \leq M_3 \sqrt{k} \, h^3 + M_4 k h^4 \quad ,$$

$$\leq \hat{C}_1 \sqrt{k} \, h^3 \quad , \qquad\qquad k = 1, 2, \cdots \, ,$$

where $\hat{C}_1 = M_3 + M_4$.

Proof of (b): Subtraction of (4.20) from (4.20) with k replaced by $k + 1$, then by straightforward estimation, and noting that $hk \leq 1$, it is not difficult to show that for $k = 1, 2, \cdots$

$$(4.22) \qquad |\hat{R}_{k+1} - \hat{R}_k| \leq |\hat{A}_{k+1}^{(1)} - \hat{A}_k^{(1)}| + |\hat{A}_{k+1}^{(2)} - \hat{A}_k^{(2)}|$$

$$\leq (13 M_3 + M_4) \frac{h^3}{\sqrt{k}} + 4 M_4 \sqrt{k} \, h^4$$

$$\leq \hat{C}_2 \frac{h^3}{\sqrt{k}} \quad ,$$

where $\hat{C}_2 = 13 M_3 + 5 M_4$.

Proof of (c): From (4.11), using (4.21), (4.22), and (2.8) we obtain for $k = 1, 2, \cdots$

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$$|\hat{b}_k| \leq \frac{\sqrt{k+2}}{(k+1)w_{k+1,k+1}} \left[(k+1)|\hat{R}_{k+1} - \hat{R}_k| + |\hat{R}_k| \right] \quad ,$$

$$\leq \hat{C}_3 \frac{h^2}{\sqrt{k}} \quad ,$$

where $\hat{C}_3 = \frac{3}{2}(\hat{C}_1 + \hat{C}_2)$.

5. <u>A Numerical Example</u>. The quadratic spline method described in Section 2 was applied to the following Abel integral equation:

$$\int_0^x \frac{1}{\sqrt{x^2-s^2}} \, y(s) \, ds = \frac{\pi}{2} J_0(x) \quad .$$

The exact solution is $y(x) = \cos(x)$.

In Table 5.1(a) and Table 5.1(b) we list the error $\epsilon(x)$, $\epsilon'(x)$ and $\epsilon''(x)$ at knots and at mid-points between the knots, respectively on $[0,3]$ for different step sizes h. The error $\epsilon(x)$, $\epsilon'(x)$ and $\epsilon''(x)$ satisfy the predicted $h^{5/2}$, h^2, and h dependence, respectively. Note also in Table 5.1(b) that the errors $\epsilon''(x)$ at the mid-points are actually $O(h^2)$ although we have not proved that this will be the case.

In Table 5.2 we list the actual error (column 2) of the derivative of our approximate solution, i.e. $\epsilon'(x)$, and its theoretical estimate (column 3) computed from equation (4.1) at knots on $[0,3]$ for $h = 0.01$.

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Table 5.1(a)

Error at Knots (h = 0.1)

x	ε(x)			ε'(x)			ε"(x)		
	h	h/3	h/9	h	h/3	h/9	h	h/3	h/9
0.0	.0000E 0	.0000E 0	.0000E 0	.0000E 0	.0000E 0	.0000E 0	-.1249E-2	-.1389E-3	-.1543E-4
0.5	-.2223E-4	-.1339E-5	-.8359E-7	.3388E-3	.4062E-4	.4693E-5	-.2474E-1	-.8074E-2	-.2673E-2
1.0	-.5565E-4	-.3438E-5	-.2173E-6	.6328E-3	.7355E-4	.8378E-5	-.4251E-1	-.1407E-1	-.4680E-2
1.5	-.8802E-4	-.5502E-5	-.3496E-6	.7744E-3	.8867E-4	.1002E-4	-.4989E-1	-.1663E-1	-.5542E-2
2.0	-.1109E-3	-.6984E-5	-.4451E-6	.7264E-3	.8209E-4	.9218E-5	-.4506E-1	-.1511E-1	-.5047E-2
2.5	-.1190E-3	-.7535E-5	-.4812E-6	.4994E-3	.5536E-4	.6151E-5	-.2919E-1	-.9896E-2	-.3316E-2
3.0	-.1110E-3	-.7059E-5	-.4515E-6	.1487E-3	.1498E-4	.1573E-5	-.6178E-2	-.2257E-2	-.7736E-3

Table 5.1(b)

Error at Mid-points (h = 0.1)

x	ε(x)			ε'(x)			ε"(x)		
	h	h/3	h/9	h	h/3	h/9	h	h/3	h/9
0.45	-.1994E-4	-.1162E-5	-.7194E-7	-.2399E-3	-.2377E-4	-.2468E-5	.3286E-3	.3862E-4	.4430E-5
0.95	-.5265E-4	-.3225E-5	-.2034E-6	-.4080E-3	-.4203E-4	-.4464E-5	.2437E-3	.2687E-4	.2985E-5
1.45	-.8519E-4	-.5316E-5	-.3376E-6	-.4730E-3	-.4977E-4	-.5351E-5	.8640E-4	.7789E-5	.7593E-6
1.95	-.1089E-3	-.6870E-5	-.4379E-6	-.4221E-3	-.4530E-4	-.4927E-5	-.9547E-4	-.1343E-4	-.1668E-5
2.45	-.1183E-3	-.7519E-5	-.4805E-6	-.2689E-3	-.2979E-4	-.3299E-5	-.2553E-3	-.3146E-4	-.3693E-5
2.95	-.1117E-3	-.7141E-5	-.4571E-6	-.5116E-4	-.7083E-5	-.8701E-6	-.3529E-3	-.4181E-4	-.4815E-5

Table 5.2

$\epsilon'(x)$ and its Theoretical Estimate

at Knots with h = 0.01

x	$\epsilon'(x)$	Theoretical Estimate
0.5	.3811E-5	.3995E-5
1.0	.6797E-5	.7012E-5
1.5	.8130E-5	.8312E-5
2.0	.7473E-5	.7577E-5
2.5	.4983E-5	.4987E-5
3.0	.1269E-5	.1176E-5

6. <u>Conclusion</u>. A global approximation method of order 2.5 to the solution of the Abel integral equation (1.1) by using a quadratic spline in C^1 has been presented.

(1) The method is self-starting and the step size h can be changed at any knot of the quadratic spline without added complication.

(2) It is a simple higher-order method that requires a comparatively small computational effort. At each step, only one equation has to be solved, whereas other higher-order methods (see, for example, [4] and [11]) require solution of a system of equations whose coefficients involve a much larger number of integrals to be evaluated.

(3) Convergence for the approximate solution as well as its first two derivatives is obtained by our method, the orders of convergence being $5/2$, 2 and 1, respectively. In addition, an asymptotic error estimate is derived for the derivative of the approximate solution.

(4) The method is economical when the values of the solution and its derivatives are required at a large number of points where usual discrete methods of computation will be time consuming.

(5) The method should be particularly useful if the derivative of $f(x)$ which is on the right-hand side of (1.1) can be computed analytically and is simple. If the derivative of $f(x)$ cannot be evaluated analytically in a convenient way, or if it is given in tabular form, one can use appropriate finite differences to perform the approximation. This, as one can show easily, does not affect the convergence of our method.

(6) The results developed in this paper give some idea of the necessary tools and the possible results for other higher-order spline approximation to the solution of some more general Abel integral equations, e.g., an approximate solution can be obtained involving cubic splines in C^2 by using the equation resulting from differentiating the original equation twice.

APPENDIX

Divergence of Methods Applying to Equation (1.1)

By Means of Splines of Order ≥ 2 with Full Continuity

For illustration, apply the method by using a quadratic spline in $C^1[0,1]$ to the following Abel integral equation:

$$\int_0^x \frac{1}{\sqrt{x^2-s^2}} \, y(s) \, ds = \frac{2}{3} x^3$$

whose exact solution is simply $y(x) = x^3$. The divergence of the numerical results can be seen very clearly in Table A.

Table A

Divergence of Numerical Results

By the Quadratic Spline Method Based on Using Equation (1.1) alone

$(h = 0.1)$

x	$\epsilon(x)$		$\epsilon'(x)$		$\epsilon''(x)$	
	h	h/3	h	h/3	h	h/3
0.0	.0000E+0	.0000E+0	.0000E+0	.0000E+0	-.1698E+0	-.5659E-1
0.2	-.3016E-3	-.2168E-3	-.1208E-1	-.3631E-1	.2311E+0	.3317E+1
0.4	-.1338E-2	-.1784E-1	-.7070E-1	-.3044E+1	.2043E+1	.2816E+3
0.6	-.5853E-2	-.1456E+1	-.3268E+0	-.2488E+3	.9952E+1	.2300E+5
0.8	-.2550E-1	-.1184E+3	-.1443E+1	-.2024E+5	.4440E+2	.1871E+7
1.0	-.1109E+0	-.9606E+4	-.6299E+1	-.1643E+7	.1942E+3	.1518E+9

$\epsilon(x)$ = Discretization error function.

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REFERENCES

[1] R. S. Anderssen, "Application and numerical solution of Abel-type integral equations",
     MRC Technical Summary Report #1787, University of Wisconsin-Madison, 1977.

[2] K. E. Atkinson, "The numerical solution of an Abel integral equation by a product
     trapezoidal method", Siam J. Num. Anal., 11 (1974), 97-101.

[3] M. P. Benson, "Errors in numerical quadrature for certain singular integrals and the
     numerical solution of Abel integral equations", Ph.D. Thesis, University of
     Wisconsin-Madison, 1973.

[4] H. Brunner, "The numerical solution of a class of Abel integral equations by piecewise
     polynomials", J. Comp. Phys. 12 (1973), 412-416.

[5] H. Brunner, "Global solution of the generalized Abel integral equation by implicit
     interpolation", Math. Comp. 28 (1974), 61-67.

[6] E. Edels, K. Hearne, and A. Young, "Numerical solution of the Abel integral equation",
     J. Math. and Phys. 41 (1962), 62-75.

[7] P. A. W. Holyhead and S. McKee, "Linear multistep methods for the generalized Abel
     integral equation", submitted for publication.

[8] H. S. Hung, "Error analysis of a linear spline method for solving an Abel integral
     equation", MRC Technical Summary Report #1904, University of Wisconsin-Madison,
     1978.

[9] L. V. Kantorovich and V. I. Krylov, "Approximate methods of higher analysis", 3rd ed.,
     GITTL, Moscow, 1950; English transl., Interscience, New York; Noordhoff, Groningen,
     1958.

[10] P. Linz, "The numerical solution of Volterra integral equations by finite difference
     methods", MRC Technical Summary Report #825, University of Wisconsin-Madison, 1967.

[11] P. Linz, "Application of Abel transforms to the numerical solution of problems in
     electrostatics and elasticity", MRC Technical Summary Report #826, University of
     Wisconsin-Madison, 1967.

[12] P. Linz and B. Noble, "A numerical method for treating indentation problems", J. Engrg. Math, 5 (1971) 227-231.

[13] B. Noble, "Methods for solving integral equations", author listing and subject listing, MRC Technical Summary Reports #1176 and #1177, University of Wisconsin-Madison, 1971.

[14] R. Weiss, "Product integration for the generalized Abel equations", Math. Comp. 26 (1972), 177-190.

[15] R. Weiss and R. S. Anderssen, "A product integration method for a class of singular first kind Volterra equations", Numer. Math. 18 (1972), 442-456.

[16] A. Young, "The application of approximate product integration to the numerical solution of integral equations", Proc. Roy. Soc. London Ser. A 224 (1954), 561-573.

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 1921	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE *(and Subtitle)* A Higher Order Global Approximation Method for Solving an Abel Integral Equation by Quadratic Splines		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Hing-Sum Hung		8. CONTRACT OR GRANT NUMBER(s) DAAG29-75-C-0024
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 7- Numerical Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P. O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE February 1979
		13. NUMBER OF PAGES 21
14. MONITORING AGENCY NAME & ADDRESS*(If different from Controlling Office)* MRC-TSR-1921		15. SECURITY CLASS. *(of this report)* UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT *(of this Report)*

Approved for public release; distribution unlimited. 26 p.

17. DISTRIBUTION STATEMENT *(of the abstract entered in Block 20, if different from Report)*

Technical summary rept.,

18. SUPPLEMENTARY NOTES

19. KEY WORDS *(Continue on reverse side if necessary and identify by block number)*

Abel integral equation, Global approximate solution, Quadratic splines, Higher order methods, Asymptotic error estimates, Product integration.

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

A quadratic spline approximation in C^1 to the solution of the Abel integral equation

$$\int_0^x \frac{1}{\sqrt{x^2-s^2}}\, y(s)\, ds = f(x)\ , \qquad x \geq 0\ ,$$

is constructed. It is shown that if y''' is Lipschitz continuous, then the approximation and its first two derivatives converge to the corresponding exact solutions at each point in the interval of integration, the orders of convergence being two, two and one, respectively. In addition, if $y \in C^4[0,1]$, then

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ABSTRACT (continued)

an asymptotic error estimate for the derivative of the approximate solution is obtained, and convergence of the approximate solution is proved to be of order $h^{5/2}$. The method is illustrated by a numerical example.